

Hypersurfaces and Codazzi tensors

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Abstract

In this paper we deal with the following problem: Let (M^n, \langle, \rangle) be an n -dimensional Riemannian manifold and $f : (M^n, \langle, \rangle) \rightarrow \mathbb{R}^{n+1}$ an isometric immersion. Find all Riemannian metrics on M^n that can be realized isometrically as immersed hypersurfaces in \mathbb{R}^{n+1} . More precisely, given another Riemannian metric $\widetilde{\langle, \rangle}$ on M^n , find necessary and sufficient conditions such that the Riemannian manifold $(M^n, \widetilde{\langle, \rangle})$ admits an isometric immersion \tilde{f} into the Euclidean space \mathbb{R}^{n+1} . If such an isometric immersion exists, how can one describe \tilde{f} in terms of f ?

1 Introduction and statement of results

According to a fundamental result due to Nash [6], every n -dimensional Riemannian manifold (M^n, \langle, \rangle) admits an isometric immersion into the Euclidean space \mathbb{R}^{n+m} , for some large m . It is therefore meaningful to ask for isometric immersions of (M^n, \langle, \rangle) into \mathbb{R}^{n+m} with the lowest possible codimension m . Along this line, Schlaefli in 1873 and later Yau [7] posed the following conjecture: *any 2-dimensional Riemannian manifold always has a local isometric immersion into \mathbb{R}^3 .*

It is not always possible for a Riemannian manifold to be realized isometrically as a hypersurface in a Euclidean space. For example, it is well known that the hyperbolic space \mathbb{H}^n does not admit an isometric immersion into \mathbb{R}^{n+1} , even locally for $n \geq 3$. In [8], Vilms by using the method of bivectors gave necessary and sufficient conditions for the existence of local isometric immersions of (M^n, \langle, \rangle) into the Euclidean space \mathbb{R}^{n+1} . Barbosa, do Carmo and Dajzcer [1, 2] gave necessary and sufficient conditions for a Riemannian manifold to be minimally immersed as a hypersurface in a space form.

The aim of this paper is to begin the study of and call attention to the following problem: Let (M^n, \langle, \rangle) be an n -dimensional Riemannian manifold and $f : (M^n, \langle, \rangle) \rightarrow \mathbb{R}^{n+1}$ an isometric immersion. Find all Riemannian metrics on M^n that can be realized isometrically as immersed hypersurfaces in \mathbb{R}^{n+1} . More precisely, given another Riemannian metric $\widetilde{\langle, \rangle}$ on M^n , find necessary

and sufficient conditions such that the Riemannian manifold $(M^n, \widetilde{\langle, \rangle})$ admits an isometric immersion \widetilde{f} into the Euclidean space \mathbb{R}^{n+1} . If such an isometric immersion exists, how can one describe \widetilde{f} in terms of f ?

Any Riemannian metric \langle, \rangle on M^n determines uniquely a non-singular $(1,1)$ -tensor field L which is positive definite, self-adjoint with respect to \langle, \rangle and satisfies $\widetilde{\langle X, Y \rangle} = \langle LX, Y \rangle$, for arbitrary tangent vector fields X, Y . Conversely, every positive definite $(1,1)$ -tensor field L which is self-adjoint with respect to \langle, \rangle gives rise to a new Riemannian metric on M^n .

In case $L = Id$, where Id is the identity map, the above problem reduces to a rigidity question about f . In fact, according to the classical Beez-Killing theorem, any other isometric immersion $\widetilde{f} : (M^n, \langle, \rangle) \rightarrow \mathbb{R}^{n+1}$ coincides with f up to an isometry of \mathbb{R}^{n+1} , provided that the rank of the shape operator of f is at least three.

When $L = e^\varphi Id$, where φ is a smooth function on M^n , the above problem reduces to the study of conformally deformable hypersurfaces. This case has been studied in details by Cartan [3]. A modern version of Cartan's results was given recently by Dajczer and Tojeiro [4].

An important class of Riemannian metrics on M^n arises in the case where $L = Q^2$, Q being a non-singular Codazzi tensor.

In this paper, we give a complete answer to the above problem for this class of Riemannian metrics.

Theorem 1 *Let $f : (M^n, \langle, \rangle) \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold (M^n, \langle, \rangle) with shape operator A associated with a unit normal vector field N . Let $\widetilde{\langle, \rangle}$ be a new metric on M^n given by $\widetilde{\langle X, Y \rangle} = \langle Q^2 X, Y \rangle$, where Q is a non-singular Codazzi tensor and X, Y are arbitrary tangent vector fields. Assume that $\text{rank } A \geq 3$. Then:*

(i) *The Riemannian manifold $(M^n, \widetilde{\langle, \rangle})$ admits an isometric immersion into \mathbb{R}^{n+1} if and only if Q commutes with A . If $\widetilde{f} : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$ is such an isometric immersion, then \widetilde{f} is rigid with shape operator $\widetilde{A} = \pm Q^{-1} \circ A$.*

(ii) *If Q commutes with A , then there exist smooth functions $g, h : M^n \rightarrow \mathbb{R}$ such that $A(\text{grad } g) = -\text{grad } h$ and $QX = \nabla_X \text{grad } g - hAX$, where X is an arbitrary tangent vector field and ∇ stands for the Levi-Civita connection of (M^n, \langle, \rangle) . Moreover, any isometric immersion $\widetilde{f} : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$ is given by $\widetilde{f} = \tau \circ f$, where τ is an isometry of \mathbb{R}^{n+1} and $F = df(\text{grad } g) + hN$.*

2 Preliminaries

Let (M^n, \langle, \rangle) be an n -dimensional Riemannian manifold and $f : M^n \rightarrow \mathbb{R}^{n+1}$ an isometric immersion into the Euclidean space \mathbb{R}^{n+1} with induced bundle $f^*(T\mathbb{R}^{n+1})$. If N is a unit normal vector field, then any section Z of the bundle $f^*(T\mathbb{R}^{n+1})$ decomposes into a tangent and a normal component, namely

$$Z = df(Z_\top) + hN,$$

where Z_\top is a tangent vector field and h is a smooth function on M^n . Denote by $\bar{\nabla}$ the connection of the induced bundle $f^*(T\mathbb{R}^{n+1})$ arising from the usual connection in \mathbb{R}^{n+1} . For tangents vector fields X, Y of M^n , we have the Weingarten formula

$$\bar{\nabla}_X N = -df(AX)$$

and the Gauss formula

$$\bar{\nabla}_X df(Y) = df(\nabla_X Y) + \langle AX, Y \rangle N,$$

where A is a self-adjoint $(1,1)$ -tensor field known as the shape operator associated with N , and ∇ is the Levi-Civita connection of \langle, \rangle .

Using the Gauss and Weingarten formulas, one can derive the equations of Gauss and Codazzi which are, respectively,

$$R(X, Y)Z = \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,$$

$$(\nabla_X A)Y = (\nabla_Y A)X,$$

for any tangent vector fields X, Y, Z , where R is the curvature tensor of (M^n, \langle, \rangle) and $\nabla_X A$ is the covariant derivative of A .

Conversely, the fundamental theorem of hypersurfaces ensures that if there exists a self-adjoint $(1,1)$ -tensor field A on a simply connected Riemannian manifold (M^n, \langle, \rangle) that fulfils the Gauss and the Codazzi equations, then (M^n, \langle, \rangle) admits an isometric immersion into the Euclidean space \mathbb{R}^{n+1} with shape operator A .

A self-adjoint $(1,1)$ -tensor field Q on the Riemannian manifold (M^n, \langle, \rangle) is said to be a Codazzi tensor if it satisfies the Codazzi equation $(\nabla_X Q)Y = (\nabla_Y Q)X$ for arbitrary tangent vector fields X, Y . A family of Codazzi tensors on (M^n, \langle, \rangle) is given by $Q = Id - tA, t \in \mathbb{R}$. Moreover, if $g, h : M^n \rightarrow \mathbb{R}$ are smooth functions that satisfy $A(\text{grad } g) = -\text{grad } h$, then the $(1,1)$ -tensor field Q defined by $QX = \nabla_X \text{grad } g - hAX$ is a Codazzi tensor, where X is an arbitrary tangent vector field. In fact, this follows by a direct computation and the Gauss equation for f .

Consider now a new metric $\widetilde{\langle, \rangle}$ on M^n given by $\widetilde{\langle X, Y \rangle} = \langle Q^2 X, Y \rangle$, where Q is a non-singular Codazzi tensor. We recall the well known relation between the metric $\widetilde{\langle, \rangle}$ and the corresponding Levi-Civita connection $\widetilde{\nabla}$, namely

$$\begin{aligned} 2\widetilde{\langle \widetilde{\nabla}_Y X, Z \rangle} &= X\widetilde{\langle Y, Z \rangle} + Y\widetilde{\langle X, Z \rangle} - Z\widetilde{\langle X, Y \rangle} - \widetilde{\langle [X, Z], Y \rangle} \\ &\quad - \widetilde{\langle [Y, Z], X \rangle} - \widetilde{\langle [X, Y], Z \rangle}. \end{aligned}$$

In view of the Codazzi equation $(\nabla_X Q)Y = (\nabla_Y Q)X$, a direct computation yields

$$\widetilde{\nabla}_X Y = Q^{-1}(\nabla_X(QY)). \quad (2.1)$$

Moreover, the curvature tensor \widetilde{R} of $(M^n, \widetilde{\langle, \rangle})$ is given by

$$\widetilde{R}(X, Y)Z = Q^{-1}(R(X, Y)QZ) \quad (2.2)$$

for arbitrary tangent vector fields X, Y, Z .

3 Proof of the result

For the proof of the main result we need the following auxiliary results.

Proposition 2 *Let $f : (M^n, \langle, \rangle) \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold (M^n, \langle, \rangle) with shape operator A associated with a unit normal vector field N . Let $\widetilde{\langle, \rangle}$ be a new metric on M^n given by $\widetilde{\langle X, Y \rangle} = \langle Q^2 X, Y \rangle$, where Q is a non-singular Codazzi tensor and X, Y are arbitrary tangent vector fields. Assume that $\text{rank } A \geq 3$. Then the Riemannian manifold $(M^n, \widetilde{\langle, \rangle})$ admits an isometric immersion into \mathbb{R}^{n+1} if and only if Q commutes with A . If $\tilde{f} : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$ is such an isometric immersion, then \tilde{f} is rigid with shape operator $\tilde{A} = \pm Q^{-1} \circ A$.*

Proof. We assume that there exists an isometric immersion $\tilde{f} : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$ with shape operator \tilde{A} . In view of (2.2), the Gauss equation for \tilde{f} yields

$$\begin{aligned} & \langle Q \circ \tilde{A}(Y), QZ \rangle Q \circ \tilde{A}(X) - \langle Q \circ \tilde{A}(Y), QZ \rangle Q \circ \tilde{A}(X) \\ &= \langle AY, QZ \rangle AX - \langle AX, QZ \rangle AY \end{aligned}$$

for any tangent vector fields X, Y, Z , or equivalently

$$Q \circ \tilde{A}(X) \wedge Q \circ \tilde{A}(Y) = AX \wedge AY, \quad (3.1)$$

where \wedge stands for the wedge product.

We claim that $\ker A = \ker \tilde{A}$. Let e_1, \dots, e_r be an orthonormal basis of $(\ker A)^\perp$ with respect to \langle, \rangle , such that $Ae_i = k_i e_i, i = 1, \dots, r$, where $r = \dim(\ker A)^\perp \geq 3$ and $(\ker A)^\perp$ stands for the orthogonal complement of the kernel of A . From (3.1), we get $AX \wedge e_i = 0$, for any $X \in \ker \tilde{A}$ and $i = 1, \dots, r$. Thus $X \in \ker A$ and $\ker \tilde{A} \subseteq \ker A$. Conversely, let $X \in \ker A$. Then (3.1) yields $Q \circ \tilde{A}(X) \wedge Q \circ \tilde{A}(e_i) = 0$, for $i = 1, \dots, r$. Since $Q \circ \tilde{A}(e_i) \neq 0$, we obtain $Q \circ \tilde{A}(X) = \rho_i Q \circ \tilde{A}(e_i)$, for some $\rho_i, i = 1, \dots, r$, or equivalently, $\tilde{A}(X - \rho_i e_i) = 0$. In view of $\ker \tilde{A} \subseteq \ker A$, we get $AX = \rho_i Ae_i$ for $i = 1, \dots, r$. This implies that $\rho_i = 0$ for all $i = 1, \dots, r$. Thus $Q \circ \tilde{A}(X) = 0$ and $X \in \ker \tilde{A}$. Consequently $\ker A = \ker \tilde{A}$.

Let $X \in (\ker A)^\perp$ and assume that $Q \circ \tilde{A}(X), AX$ are linearly independent. Since $\dim(\ker A)^\perp \geq 3$, there exists $Y \in (\ker A)^\perp$ such that $Q \circ \tilde{A}(X), AX, AY$ are linearly independent. Then from (3.1) we obtain

$$Q \circ \tilde{A}(X) \wedge Q \circ \tilde{A}(X) \wedge Q \circ \tilde{A}(Y) = Q \circ \tilde{A}(X) \wedge AX \wedge AY \neq 0,$$

which is contradiction. Thus $Q \circ \tilde{A}(X), AX$ are linearly dependent for any $X \in (\ker A)^\perp$ and consequently $Q \circ \tilde{A}(X) = a(X)AX$, where $a(X)$ depends on X .

We choose an arbitrary basis X_1, \dots, X_r of $(\text{Ker } A)^\perp$. Then we have $Q \circ \tilde{A}(X_i) = a(X_i)AX_i, i = 1, \dots, r$. Since $X_i + X_j \in (\text{Ker } A)^\perp$, we have $Q \circ \tilde{A}(X_i + X_j) = a(X_i + X_j)A(X_i + X_j)$, for any $i, j = 1, \dots, r$, and consequently,

$$\left(a(X_i + X_j) - a(X_i)\right)AX_i + \left(a(X_i + X_j) - a(X_j)\right)AX_j = 0.$$

This means that $a(X_i + X_j) = a(X_i) = a(X_j)$, for all $i, j = 1, \dots, r$. Moreover, for any real number λ , using the linearity of $Q \circ \tilde{A}$, we obtain $Q \circ \tilde{A}(\lambda X) = \lambda Q \circ \tilde{A}(X) = \lambda a(X)AX$. Because of $Q \circ \tilde{A}(\lambda X) = a(\lambda X)A(\lambda X) = \lambda a(\lambda X)A(X)$, we conclude $a(\lambda X) = a(X)$. Therefore, there exists a constant a such that $Q \circ \tilde{A}(X) = aA(X)$ for any X . Appealing to (3.1) we get $a = \pm 1$ and therefore $\tilde{A} = \pm Q^{-1} \circ A$. Since \tilde{A} is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, $Q^2 \circ \tilde{A}$ is self-adjoint with respect to the metric $\langle \cdot, \cdot \rangle$, and so we obtain $Q \circ A = A \circ Q$.

Conversely, we assume that Q commutes with A . Then $\tilde{A} := \pm Q^{-1} \circ A$ is self-adjoint with respect to the metric $\langle \cdot, \cdot \rangle$. Moreover, in view of (2.2), \tilde{A} satisfies the Gauss equation. In addition, it can be easily seen using (2.1) that \tilde{A} is a Codazzi tensor. Then according to the fundamental theorem of hypersurfaces there exists an isometric immersion $\tilde{f} : (M^n, \langle \cdot, \cdot \rangle) \rightarrow \mathbb{R}^{n+1}$ with shape operator $\tilde{A} = \pm Q^{-1} \circ A$. Obviously, due to the Beez-Killing theorem \tilde{f} is rigid and the proof is complete. ■

The following result is essentially due to Dajczer and Tojeiro [5].

Proposition 3 *Let $f : (M^n, \langle \cdot, \cdot \rangle) \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold $(M^n, \langle \cdot, \cdot \rangle)$ with shape operator A associated with a unit normal vector field N . Assume that Q is a non-singular Codazzi tensor which commutes with A . Then there exist:*

- (i) *an immersion $F : M^n \rightarrow \mathbb{R}^{n+1}$ such that $dF = df \circ Q$, and*
- (ii) *smooth functions $g, h : M^n \rightarrow \mathbb{R}$ such that $A(\text{grad } g) = -\text{grad } h$, $QX = \nabla_X \text{grad } g - hAX$, where X is an arbitrary tangent vector field and F is given by $F = df(\text{grad } g) + hN$.*

Furthermore, F is an isometric immersion of the Riemannian manifold $(M^n, \langle \cdot, \cdot \rangle)$ into \mathbb{R}^{n+1} , where the metric $\langle \cdot, \cdot \rangle$ on M^n is given by $\langle \tilde{X}, \tilde{Y} \rangle = \langle Q^2 X, Y \rangle$, X, Y being arbitrary tangent vector fields.

Proof. (i) Let e_1, \dots, e_{n+1} be the standard orthonormal basis of \mathbb{R}^{n+1} . We consider the 1-forms $\omega_i := \langle df \circ Q, e_i \rangle, i = 1, \dots, n+1$. Then for any tangent vector fields X, Y , we get

$$\begin{aligned} d\omega_i(X, Y) &= X(\omega_i(Y)) - Y(\omega_i(X)) - \omega_i([X, Y]) \\ &= \langle \nabla_X df(QY), e_i \rangle - \langle \nabla_Y df(QX), e_i \rangle - \langle df(Q[X, Y]), e_i \rangle. \end{aligned}$$

Using the Gauss formula, we obtain

$$\begin{aligned} d\omega_i(X, Y) &= \langle df(\nabla_X(QY) - \nabla_Y(QX) - Q([X, Y])), e_i \rangle \\ &\quad + \langle [Q, A]X, Y \rangle \langle N, e_i \rangle, \end{aligned}$$

for each $i = 1, \dots, n+1$, where $[Q, A] = Q \circ A - A \circ Q$. Since Q is a Codazzi tensor and commutes with A , we deduce that ω_i 's are closed. Thus there exist functions $u_i : M^n \rightarrow \mathbb{R}$ such that $\omega_i = du_i$ $i = 1, \dots, n+1$. Then the map $F : M^n \rightarrow \mathbb{R}^{n+1}$ given by $F := \sum_{i=1}^{n+1} u_i e_i$ clearly satisfies $dF = df \circ Q$.

(ii) We view the immersion F introduced in (i) as a section of the induced bundle $f^*(T\mathbb{R}^{n+1})$. Decomposing into a tangent and a normal component, we get

$$F = df(Z) + hN, \quad (3.2)$$

where Z is a tangent vector field and h is a smooth function on M^n . Then on account of Gauss and Weingarten formulas, for any tangent vector field X , we have

$$dF(X) = df(\nabla_X Z - hAX) + (\langle AX, Z \rangle + Xh)N,$$

or equivalently

$$df(QX - \nabla_X Z + hAX) = (\langle AX, Z \rangle + Xh)N.$$

From this we find that $QX = \nabla_X Z - hAX$ and $AZ = -\text{grad } h$. Furthermore, we deduce that the 1-form ω defined by $\omega(X) = \langle X, Z \rangle$ is closed. Since M^n is simply connected, there exists a function $g : M^n \rightarrow \mathbb{R}$ such that $\omega = dg$, or equivalently $Z = \text{grad } g$. Then (3.2) is written as $F = df(\text{grad } g) + hN$.

Moreover, the metric $\widetilde{\langle, \rangle}$ induced on M^n is given by

$$\widetilde{\langle X, Y \rangle} = \langle dF(X), dF(Y) \rangle = \langle df(QX), df(QY) \rangle = \langle Q^2 X, Y \rangle,$$

for any tangent vector fields X, Y . This completes the proof. ■

Proof of the Theorem 1. Let $f : (M^n, \langle, \rangle) \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of the simply connected Riemannian manifold (M^n, \langle, \rangle) with shape operator A associated with a unit normal vector field N . We consider the Riemannian metric $\widetilde{\langle, \rangle}$ on M^n given by $\widetilde{\langle X, Y \rangle} = \langle Q^2 X, Y \rangle$, where Q is a non-singular Codazzi tensor and X, Y are arbitrary tangent vector fields.

(i) According to Proposition 2, the Riemannian manifold $(M^n, \widetilde{\langle, \rangle})$ admits an isometric immersion into \mathbb{R}^{n+1} if and only if Q commutes with A . Moreover, such an isometric immersion $\widetilde{f} : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$ is rigid with shape operator $\widetilde{A} = \pm Q^{-1} \circ A$.

(ii) If Q commutes with A , then Proposition 3 implies that there exists an isometric immersion $F : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$. Moreover, there exist smooth functions $g, h : M^n \rightarrow \mathbb{R}$ such that $A(\text{grad } g) = -\text{grad } h$, $QX = \nabla_X \text{grad } g - hAX$, where X is an arbitrary tangent vector field, and $F = df(\text{grad } g) + hN$. In addition, since $\text{rank } A \geq 3$, by the Beez-Killing theorem, any isometric immersion $\widetilde{f} : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$ is given by $\widetilde{f} = \tau \circ F$, where τ is an isometry of \mathbb{R}^{n+1} . ■

Remark 4 It is worth noticing that f and F in Proposition 3 have the same Gauss map, and consequently the immersions f and \widetilde{f} in Theorem 1 have congruent Gauss maps.

The following two examples illustrate the main result.

Example 5 Let $f : (M^n, \langle, \rangle) \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold (M^n, \langle, \rangle) with shape operator A corresponding to a unit normal vector field N . We consider the tensor field $Q := Id - tA$, $t \in \mathbb{R}$, where t is chosen such that Q is non-singular. Obviously, Q is a Codazzi tensor that commutes with A . According to our results, the Riemannian manifold $(M^n, \widetilde{\langle, \rangle})$ is realized isometrically as an immersed hypersurface in \mathbb{R}^{n+1} , where the metric $\widetilde{\langle, \rangle}$ is given by $\widetilde{\langle X, Y \rangle} = \langle Q^2 X, Y \rangle$, and X, Y are arbitrary tangent vector fields.

Viewing f as a section of the induced bundle $f^*(T\mathbb{R}^{n+1})$, and decomposing into a tangent and a normal component, we get

$$f = df(x_\top) + pN, \quad (3.3)$$

where x_\top is a tangent vector field and $p = \langle f, N \rangle$ is the support function of f . Differentiating (3.3) with respect to a tangent vector field X and using Gauss and Weingarten formulas, we obtain

$$\nabla_X x_\top = X + pAX \quad \text{and} \quad Ax_\top = -\text{grad } p.$$

Moreover we, easily, see that $\text{grad}(\frac{1}{2}|f|^2) = x_\top$. Consequently the functions $\frac{1}{2}|f|^2$ and p satisfy

$$A \text{grad}(\frac{1}{2}|f|^2) = -\text{grad } p \quad \text{and} \quad QX = \nabla_X \text{grad}(\frac{1}{2}|f|^2) - (p+t)AX.$$

So the functions $g := \frac{1}{2}|f|^2$ and $h := p+t$ can be used to construct the isometric immersion $F : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$, that is $F = df(\text{grad } g) + hN$. In view of (3.3), it follows that $F = f + tN$, a parallel hypersurface to f .

Example 6 Let $f : (M^n, \langle, \rangle) \rightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a simply connected Riemannian manifold (M^n, \langle, \rangle) with shape operator A associated with a unit normal vector field N . Consider a constant vector a in \mathbb{R}^{n+1} and view it as a section of the induced bundle $f^*(T\mathbb{R}^{n+1})$. We decompose a as

$$a = df(a_\top) + \langle N, a \rangle N. \quad (3.4)$$

Differentiating with respect to a tangent vector field X and using Gauss and Weingarten formulas, we obtain

$$\nabla_X a_\top = \langle N, a \rangle AX \quad \text{and} \quad Aa_\top = -\text{grad } \langle N, a \rangle.$$

Since $\text{grad } \langle f, a \rangle = a_\top$, we have $A \text{grad } \langle f, a \rangle = -\text{grad } \langle N, a \rangle$. So the functions $g := \langle f, a \rangle$ and $h := \langle N, a \rangle + 1$ satisfy

$$A \text{grad } g = -\text{grad } h \quad \text{and} \quad QX = \nabla_X \text{grad } g - (\langle N, a \rangle + 1)AX = -AX,$$

and can be used to construct the isometric immersion $F : (M^n, \widetilde{\langle, \rangle}) \rightarrow \mathbb{R}^{n+1}$ where $\widetilde{\langle X, Y \rangle} = \langle A^2 X, Y \rangle$, and X, Y are arbitrary tangent vector fields. In this case, we have $F = df(\text{grad } g) + hN = df(a_\top) + (\langle N, a \rangle + 1)N = a + N$, the Gauss map of f followed by a parallel translation.

In the following remark we discuss the uniqueness of the pair of functions g, h , for a given Codazzi tensor Q which commutes with the shape operator A .

Remark 7 Under the assumptions of Theorem 1, we suppose that there exist two pairs of functions $(g, h), (g_1, h_1)$, such that

$$A(\text{grad } g) = -\text{grad } h, A(\text{grad } g_1) = -\text{grad } h_1$$

and

$$QX = \nabla_X \text{grad } g - hAX = \nabla_X \text{grad } g_1 - h_1AX,$$

where X is an arbitrary tangent vector field. Then, according to Proposition 3, the immersions $F = df(\text{grad } g) + hN$ and $F_1 = df(\text{grad } g_1) + h_1N$ induce the same metric on M^n and satisfy $dF = df \circ Q = dF_1$. Thus $F_1 = F + a$, for a constant vector a . Viewing a as a section of the induced bundle $f^*(T\mathbb{R}^{n+1})$, we decompose as in (3.4). Since $F = df(\text{grad } g) + hN$ and $F_1 = df(\text{grad } g_1) + h_1N$, from (3.3) and $F_1 = F + a$, we obtain $g_1 = g + \langle f, a \rangle + c$ and $h_1 = h + \langle N, a \rangle$, where c is a real constant. Thus the pair of functions g, h are uniquely determined up to the functions $\langle f, a \rangle$ and $\langle N, a \rangle$.

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